

NONLINEAR PROJECTIVE FILTERING I: BACKGROUND IN CHAOS THEORY

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Abstract— We derive a locally projective noise reduction scheme for nonlinear time series using concepts from deterministic dynamical systems, or chaos theory. We will demonstrate its effectiveness with an example with known deterministic dynamics and discuss methods for the verification of the results in the case of an unknown deterministic system.

I. INTRODUCTION

The paradigm of deterministic chaos as an alternative explanation for complex temporal behaviour [1] has made the development of novel signal processing techniques necessary. Deterministic chaotic systems are characterised by exponentially decaying correlations and thus broad band power spectra. Thus, except for the case of highly oversampled, time continuous signals, filtering by frequency cannot be applied since signal and noise have similar spectral properties. On the other hand, deterministic dynamics of the form

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n), \quad (1)$$

or given by an ordinary differential equation of first order satisfying a Lipschitz condition for unique solutions, generates strong signatures when viewed in its phase space. Nonlinear noise reduction methods have been developed to exploit these structures. Conceptual as well as technical issues arising in such a situation have been well discussed in the literature, see Kostelich and Schreiber [2] for a review containing the relevant references.

In engineering applications, we usually face a different situation — the signals themselves often contain a stochastic component and we cannot make the assumption that the system is of the form (1) and deterministic chaos is present. It turns out that at least for a subclass of the nonlinear filtering schemes, the phase space projection techniques [3], deterministic chaos is not a necessary requirement. The only assumption that has to be made is that the signal of interest is approximately described by a manifold that has a lower dimension than some phase space it is embedded in.

This is formally true for low dimensional deterministic signals, but also for certain stochastically driven nonlinear phenomena.

In this paper, a phase space projection scheme for noise reduction will be motivated and described. We will give an example with known deterministic dynamics for the purpose of illustration. Applications to real data are discussed in Ref. [4] in this volume.

II. METHOD

Let $\{\mathbf{x}_n\}$ be the states of a system at times $n = 1, \dots, N$, represented in some vector space \mathcal{R}^m . A $(m - Q)$ -dimensional submanifold \mathcal{F} of this space can be specified by $F_q(\mathbf{y}) = 0$, $q = 1, \dots, Q$. Suppose such a manifold exists such that the sequence of vectors $\{\mathbf{x}_n\}$, possibly changed by small displacements $\{\epsilon_n\}$ lies on that surface:

$$F_q(\mathbf{x}_n + \epsilon_n) = 0, \quad \forall q, n. \quad (2)$$

The quantity $\sqrt{\langle \epsilon^2 \rangle}$ denotes the (root mean squared) average error we make by approximating the points $\{\mathbf{x}_n\}$ by the manifold \mathcal{F} . For a useful approximation we require the functions F_q to be smooth and the sequence $\{\epsilon_n\}$ to be small in the rms sense.

In a measurement, we can only obtain noisy data $\mathbf{y}_n = \mathbf{x}_n + \boldsymbol{\eta}_n$, where $\{\boldsymbol{\eta}_n\}$ is some random contamination. The manifold \mathcal{F} is not known *a priori* and has to be estimated from the data. By projecting the points $\{\mathbf{y}_n\}$ onto the estimated manifold \mathcal{F} we can aim to recover $\mathbf{x}'_n = \mathbf{x}_n + \epsilon_n$. If we can find a suitable manifold — and carry out the projections — such that $\langle \epsilon^2 \rangle < \langle \boldsymbol{\eta}^2 \rangle$, then we have in fact reduced the observational error: The projected sequence is closer to the true states $\{x_n\}$ than the noisy observations.

In all the cases discussed in this paper, only a scalar measurement of the system states is available: $s_n = s(\mathbf{x}_n)$, $n = 1, \dots, N$. A multi-dimensional vector representation can be obtained by considering also time delayed copies of the scalar sequence: $\mathbf{s}_n = (s_{n-(m-1)\tau}, s_{n-(m-2)\tau}, \dots, s_n)$. For deterministic dynamical systems, theorems are available [5, 6] on

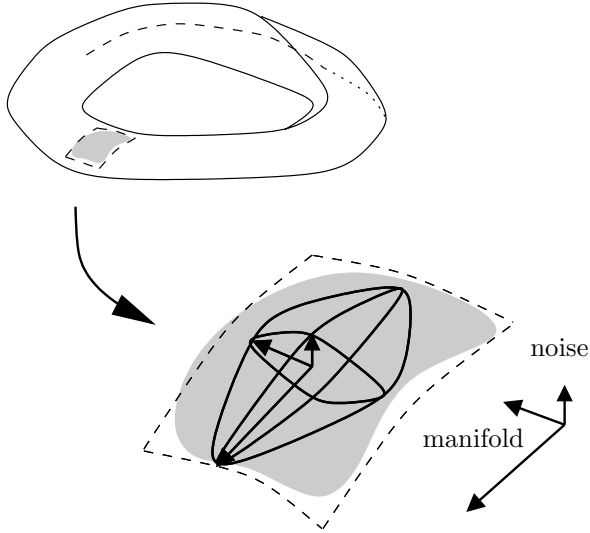


Figure 1: Illustration of the local projection scheme. For each point to be corrected, a neighbourhood is formed (grey shaded area), the point cloud in which is then approximated by an ellipsoid. An approximately two-dimensional manifold embedded in a three-dimensional space could for example be cleaned by projecting onto the first two principal directions.

the equivalence of the sequences $\{\mathbf{s}_n\}$ and $\{\mathbf{x}_n\}$. Suppose a system is governed by deterministic equations of motion in $m-1$ -dimensional delay coordinate space. Then Eq.(1) becomes

$$\mathbf{s}_n - F(\mathbf{s}_{n-(d-1)\tau}, \dots, \mathbf{s}_{n-\tau}) = 0, \quad \forall n. \quad (3)$$

This means that in m -dimensional embedding space, there exists an $m-1$ -dimensional manifold containing the signal. Noisy measurements of such processes can be cleaned by imposing the relation (3) on the data. In higher dimensional embeddings, more than one independent equations of the form (3) exist. Still, these functions F_q are unknown and have to be approximated by a fit.

In time series work, the most practical way to approximate data by a manifold is by a locally linear representation. It should in principle be possible to fit global nonlinear constraints \hat{F}_q from data, but the problem is complicated by the necessity to have Q locally independent equations. In the locally linear case this is achieved by establishing local principal components. The derivation will not be repeated here, it is carried out for example in Ref. [1]. The resulting algorithm proceeds as follows. In an embedding space of dimension m we form delay vectors \mathbf{s}_n . For each of these we construct small neighborhoods \mathcal{U}_n , so that the neighbouring points are $\mathbf{s}_k, k \in \mathcal{U}_n$. Within each

neighbourhood, we compute the local mean

$$\bar{\mathbf{s}}^{(n)} = \frac{1}{|\mathcal{U}_n|} \sum_{k \in \mathcal{U}_n} \mathbf{s}_k \quad (4)$$

and the $(m \times m)$ covariance matrix

$$C_{ij} = \frac{1}{|\mathcal{U}_n|} \sum_{k \in \mathcal{U}_n} (\mathbf{s}_k)_i (\mathbf{s}_k)_j - \bar{\mathbf{s}}_i^{(n)} \bar{\mathbf{s}}_j^{(n)}. \quad (5)$$

It has been found advantageous [3] to introduce a diagonal weight matrix R and define a transformed version of the covariance matrix $\Gamma_{ij} = R_{ii} C_{ij} R_{jj}$ for the calculation of the principal directions. In order to penalise corrections based on the first and last coordinates in the delay window one puts $R_{00} = R_{mm} = r$ where r is large. The other values on the diagonal of R are 1.

The eigenvectors of the matrix Γ_{ij} are the semi-axes of the best approximating ellipsoid of the cloud of points. These are local versions of the well known principal components, or singular vectors, see for example Refs. [7, 8]. If the clean data lives near a smooth manifold with dimension $m_0 < m$, and if the variance of the noise is sufficiently small for the linearisation to be valid, then for the noisy data the covariance matrix will have large eigenvalues spanning the smooth manifold and small eigenvalues in all other directions. Of course, this is strictly true only if the neighbourhoods are larger than the noise level. In practice, a tradeoff between the clear definition of the noise directions and a good linear approximation has to be balanced.

By projecting onto the subspace of large eigenvectors, we move the vector under consideration towards the manifold. The procedure is graphically illustrated in Fig. 1. The correction is done for each embedding vector, resulting in a set of corrected vectors in embedding space. Since each element of the scalar time series occurs in m different embedding vectors, we finally have as many different suggested corrections, of

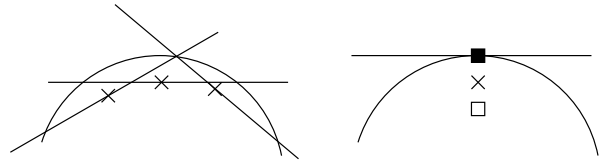


Figure 2: Local linear approximation to a one-dimensional curve. Left: approximations are not tangents but secants and all the centres of mass (crosses) of different neighbourhoods are shifted inward with respect to the curvature. Right: a tangent approximation is obtained by shifting the centre of mass outward with respect to the curvature. The open square denotes the average of the centres of mass of adjacent neighbourhoods, the filled square is the corrected centre of mass.

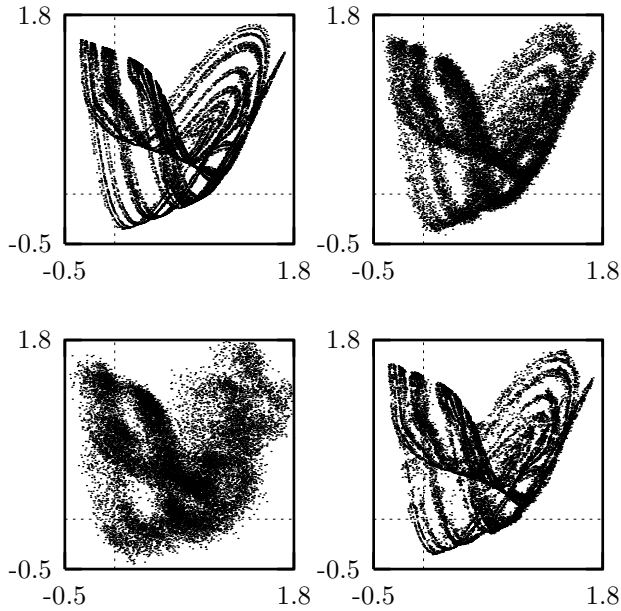


Figure 3: Ikeda map example. Upper left: noise free data; right: data contaminated with 5% additive Gaussian white noise. Lower left: data after low-pass filtering; right: data after nonlinear noise reduction.

which we simply take the average. Therefore in embedding space the corrected vectors do not precisely lie on the local subspaces but are only moved towards them.

If the local linear subspaces are determined in the way outlined above, they are not really tangent to the curved manifold but rather intersect with it, as illustrated in Fig. 2. Therefore it is preferable to use a corrected centre of mass $\bar{\mathbf{s}}^{(n)}$ given by

$$\bar{\mathbf{s}}^{(n)} = 2\bar{\mathbf{s}}^{(n)} - \frac{1}{|\mathcal{U}_n|} \sum_{k \in \mathcal{U}_n} \bar{\mathbf{s}}^{(k)} \quad (6)$$

This correction prevents a bias towards corrections in the main direction of curvature. If it is omitted, and if rather large neighbourhoods are used, the set of embedded data points in phase space may be subject to an overall contraction on multiple iterations of the procedure.

A computer program that implements the scheme described in this paper is included in the TISEAN software project and is publicly available [9].

III. NUMERICAL EXAMPLE

Let us show an example for the noise reduction capability of the algorithm for deterministically chaotic time series. The Ikeda map is given by the formula

$$z_{n+1} = 1 + 0.9z_n \exp\left(0.4i - \frac{6i}{1 + |z_n|^2}\right) \quad (7)$$

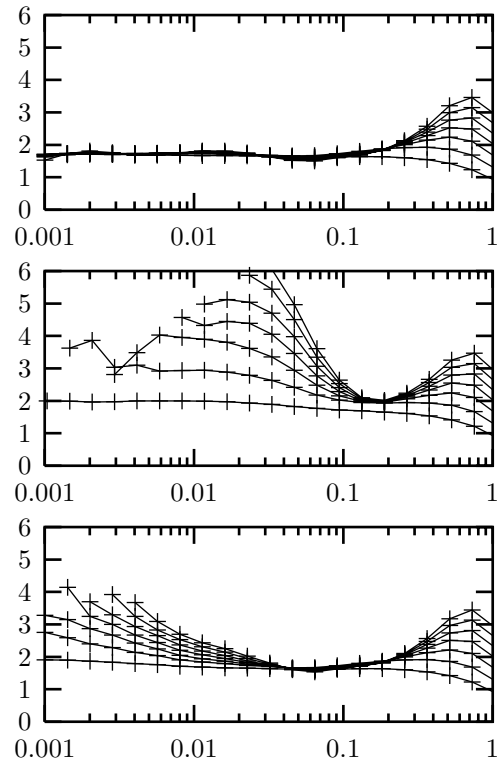


Figure 4: Local slopes of the correlation sum indicating fractal structure (Ikeda map example). Upper: noise free data. Middle: data contaminated with 5% additive Gaussian white noise. Lower: after nonlinear noise reduction. See text.

where $\{z_n\}$ is a sequence of complex numbers. Consider as a model time series the sequence $x_n = \Re(z_n)$ given by the real parts of z_n . In Fig. 3, the upper left panel shows a delay representation with delay $\tau = 1$ of the noise free sequence $\{x_n\}$. In order to mimic a simplified experimental situation, the sequence $s_n = x_n + \eta_n$ has been contaminated by adding Gaussian independent random numbers of a rms amplitude of 5% of that of the clean data, upper right panel. The lower row shows two different attempts on noise reduction. Left, a low pass filter has been applied to suppress the highest 4% of frequencies in the Nyquist interval. Since the signal has still significant power at these frequencies, the filter is inadequate and, in fact, severely distorts the phase space structure. In the right panel, nonlinear noise reduction has been applied using embeddings in $m = 7$ dimensions and local projections onto $m - Q = 3$ dimensions. The neighbourhood size was taken to be 0.02 units, which is about the absolute noise level added. The figure is the result of three iterations. The error was reduced by a factor of 1.7 in terms of rms amplitude. Note however that the data is probably much closer than that to a

true trajectory of the Ikeda system.

In Fig. 4, the effect of noise reduction on this fractal attractor is demonstrated with the help of the Grassberger-Procaccia correlation sum, $C(\epsilon)$, the fraction of pairs of points closer than ϵ in delay coordinate space. As explained for example in Ref. [1], we take the local slope in a double logarithmic plot of $C(\epsilon)$ versus ϵ as an effective scale dependent scaling exponent $d(\epsilon) = d \log C(\epsilon) / d \log \epsilon$. If a significant plateau occurs and certain precautions have been taken, the plateau value of $d(\epsilon)$ would estimate the correlation dimension of the attractor underlying the data. Indeed, such a plateau can be seen for the noise free data (upper panel) while it is rather small for the noisy data. After nonlinear noise reduction (lower panel), the scaling is recovered down to much smaller length scales.

IV. DISCUSSION

With the emergence of experiments exhibiting deterministic chaos, nonlinear filtering techniques became a necessity. The earliest approaches required detailed understanding of the dynamics and the stability structure in phase space. The phase space projection technique described in this paper goes back to Ref. [3] but some earlier techniques are quite similarly set-up [2]. These techniques do not require a detailed model, or a global fit of such a model, for the dynamics. Rather, the inhomogeneous distribution in reconstructed phase space is enhanced by projecting onto local linear subspaces. We have demonstrated in this paper that the method is effective in reducing noise in an artificial, chaotic example. Applications to real time series data will be discussed in Ref. [4] in this volume.

References

- [1] H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis*. Cambridge Univ. Press, Cambridge (1997).
- [2] E. J. Kostelich and T. Schreiber, “Noise reduction in chaotic time series data: A survey of common methods”, *Phys. Rev. E* 48 (1993) 1752.
- [3] P. Grassberger, R. Hegger, H. Kantz, C. Schaffrath, and T. Schreiber, “On noise reduction methods for chaotic data”, *Chaos* 3 (1993) 127.
- [4] T. Schreiber and H. Kantz, “Nonlinear projective filtering II: Application to real time series”, this volume (1998).
- [5] F. Takens, “Detecting Strange Attractors in Turbulence”, *Lecture Notes in Math.* Vol. 898, Springer, New York (1981).
- [6] T. Sauer, J. Yorke, and M. Casdagli, “Embedology”, *J. Stat. Phys.* 65 (1991) 579.
- [7] I. T. Jolliffe, “Principal component analysis”, Springer, New York (1986).
- [8] D. Broomhead and G. P. King, “Extracting qualitative dynamics from experimental data”, *Physica D* 20 (1986) 217.
- [9] The TISEAN software package is publicly available for download from either <http://www.mpi-pks-dresden.mpg.de/~tsa/TISEAN/docs/welcome.html> or <http://wptu38.physik.uni-wuppertal.de/Chaos/DOCS/welcome.html>.